

Perturbative classical and quantum field theory

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Abstract

In a first part we study the ϕ^{p+1} -field theory from the classical point of view. Using Butcher series we compute explicitly the perturbative expansion of the solutions and we prove that this expansion converges if the coupling constant is small enough. Then we show that we can formally recover the Heisenberg's interacting quantum field directly from this expansion. In other words we write the Heisenberg interacting field as a Butcher series.

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Introduction

Butcher series are sums indexed by planar trees introduced by J. C. Butcher [6] in order to study and classify [8] Runge Kutta methods in numerical analysis. But they appear naturally in a large class of problem (see e.g. [9], [4]) including quantum field theory (see e.g. the paper of Ch. Brouder and A. Frabetti [2], [3], [5] about Q.E.D). Moreover Ch. Brouder noticed that the structure which underlies the original Butcher's calculations [6] is exactly the Hopf algebra of rooted trees defined by D. Kreimer in his paper about renormalization [11] and he shows that finally renormalization can be seen as a Runge–Kutta method. Hence it seems that the work of J.C. Butcher and perturbative quantum field theory are closely related. In this paper we show that formally the Heisenberg's interacting quantum field can be written as a Butcher series. Hence we link the perturbative classical field theory with the perturbative quantum field theory.

Butcher series provide a precise description of the solutions of a non linear problem. In this paper we focus on ϕ^{p+1} -theory but our results can be generalized to any theory. First, we explain how Butcher series give the explicit perturbative expansion of the classical field and we prove that this expansion converges if the coupling constant is small enough. Then, we formally quantize the expansion and we show that we recover the Heisenberg picture of the interacting quantum field.

Let p be a non negative integer such that $p \geq 2$. Then consider the equation over scalar fields $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$(\square + m^2)\varphi + \lambda\varphi^p = 0 \quad (\text{K-Gp})$$

where $m > 0$ is the mass, λ is a real parameter (the coupling constant) and \square denotes the operator $\frac{\partial}{\partial(x^0)^2} - \sum_j \frac{\partial}{\partial(x^j)^2}$. We show that the solution of (K–Gp) writes as a power series indexed by planar trees.

Planar trees are rooted trees drawn into the plane with the root on the ground. The external vertices are called *leaves* and the other are called *internal vertices*. We denote by $|b|$ the number

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of internal vertices of a planar tree b . There is a unique planar tree with no internal vertices: this is the planar tree reduced to a root, let's denote by \circ this trees. We say that a planar tree is a p -tree if and only if each internal vertex has exactly p childrens. Let us denote by $\mathbb{T}(p)$ the set of p -trees. Finally given (b_1, \dots, b_p) a p -uplet of p -trees, we can define another p -tree $B_+(b_1 \dots b_p)$ by linking a new root to the roots of b_1, \dots, b_p .

Given the Cauchy datas $(\varphi^0, \varphi^1) \in H^{q+1}(\mathbb{R}^d) \times H^q(\mathbb{R}^d)$, we show (theorem 2.1) that the solution $\varphi \in \mathcal{C}^1([0, T], H^q(\mathbb{R}^d)) \cap \mathcal{C}^2([0, T], H^{q-1}(\mathbb{R}^d))$ of (K-Gp) such that $\varphi(0, \bullet) = \varphi^0$ and $\frac{\partial \varphi}{\partial t}(0, \bullet) = \varphi^1$ is given by the power series

$$\varphi = \sum_{b \in \mathbb{T}(p)} \lambda^{|b|} \phi(b) \quad (\mathcal{B})$$

where the coefficients $(\phi(b))_{b \in \mathbb{T}(p)}$ are such that

$$\begin{cases} \phi(\circ) \text{ is the solution of } (\square + m^2)\phi(\circ) = 0 \text{ with Cauchy datas } (\varphi^0, \varphi^1) \\ \phi(B_+(b_1 \dots b_p)) \text{ is the solution of } (\square + m^2)\varphi = -\phi(b_1) \dots \phi(b_p) \text{ with zero Cauchy datas} \end{cases}$$

The power series (\mathcal{B}) converges in the $\mathcal{C}^1([0, T], H^q(\mathbb{R}^d)) \cap \mathcal{C}^2([0, T], H^{q-1}(\mathbb{R}^d))$ topology if λ is small enough, it is called the *Butcher series*.

Remark

The function $\phi(\circ)$ is the (classical) free field corresponding to the Cauchy datas (φ^0, φ^1) at time $t = 0$. We can reformulate the definition of $\phi(B_+(b_1 \dots b_p))$ by

$$\phi(B_+(b_1 \dots b_p))(x) := - \int_{P^+} dy G_{ret}(x - y) \phi(b_1)(y) \dots \phi(b_p)(y)$$

where $G_{ret}(z)$ denotes the retarded Green function of the linear Klein-Gordon operator $(\square + m^2)$ and where P^+ is the half space $P^+ := \{(t, \vec{x}) \in \mathbb{R} \times \mathbb{R}^d \mid t > 0\}$.

Using the last remark we can formally quantize the Butcher series. We use the notation of the book of E. Peskin and Daniel V. Schroeder [12] p.83. i.e. we consider the operator $\phi_I(x)$ acting on Fock space defined by

$$\phi_I(t, \vec{x}) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{dk}{\sqrt{2\omega_k}} \left(\mathbf{a}_k e^{-ik \cdot x} + \mathbf{a}_k^\dagger e^{ik \cdot x} \right) \Big|_{x^0=t-t_0}$$

Here $k \cdot x$ denotes the quantity $k \cdot x := \omega_k t - \langle k, \vec{x} \rangle$ where $\langle k, \vec{x} \rangle$ denotes the usual scalar product of \mathbb{R}^d and ω_k the quantity $\omega_k := (|k|^2 + m^2)^{1/2}$. The operator $\phi_I(x)$ is the free field on space-time.

Then by analogy we define formally the family of operator $(\widehat{\phi}(b))_{b \in \mathbb{T}(p)}$ by

$$\begin{cases} \widehat{\phi}(\circ)(x) := \phi_I(x) \\ \widehat{\phi}(B_+(b_1 \dots b_p))(x) := - \int_{P^+} d^{d+1}y G_{ret}(x - y) \widehat{\phi}(b_1)(y) \dots \widehat{\phi}(b_p)(y) \end{cases}$$

Note that this definition is formal since operator product is ill defined on the diagonal.

Using the commutation rules of the $\phi_I(x)$'s, we get the following theorem:

Theorem 3.1 *for all $t > t_0$ and $\vec{x} \in \mathbb{R}^d$ we have formally*

$$\widehat{\varphi}(t, \vec{x}) := \sum_{b \in \mathbb{T}(p)} \lambda^{|b|} \widehat{\phi}(b)(x) = U^\dagger(t) \phi_I(t, \vec{x}) U(t)$$

where $U(t)$ denotes the operator

$$U(t) := \sum_{\alpha \geq 0} (-i\lambda)^\alpha \int_{t_0}^t d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \cdots \int_{t_0}^{\tau_{\alpha-1}} d\tau_\alpha H_I(\tau_1) H_I(\tau_2) \cdots H_I(\tau_\alpha)$$

and $H_I(\tau)$ is given by

$$H_I(\tau) := \frac{1}{p+1} \int_{\mathbb{R}^d} d\vec{y} \phi_I^{p+1}(\tau, \vec{y})$$

Using the time ordered product, the operator U is given by the well known formula (see [12] p.85):

$$U(t) = T \left\{ \exp \left(-i\lambda \int_{t_0}^t d\tau H_I(\tau) \right) \right\}$$

Hence we recover the Heisenberg formulation of interacting quantum field as a quantization of Butcher series. From another point of view we prove that formally the interacting quantum field given by the Heisenberg identity can be expressed as a Butcher series.

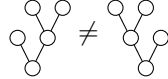
1 Planar p -trees

Let us introduce some definitions concerning planar trees.

Definition 1.1 A planar tree is an oriented connected finite graph without loop together with an embedding into the plane; we suppose that the graph has a particular node that no edge points to; this node is called the root of the tree. The root is drawn at the bottom of the tree.

Remark 1.1

The set of planar trees differs from the set of *rooted trees* used by D. Kreimer and A. Connes in their work about renormalization [11], [7]. For instance the following planar trees are different



although they represent the same rooted tree. All the results of this paper can be expressed using rooted trees instead of planar trees: we just have to adapt the definitions and add a symmetry factor in front of each term of the perturbative expansions. We prefer to work with planar trees because they permit to avoid these symmetry factors and to get simpler formulae.

Notation 1.1

Let b be a planar tree

1. The external vertices of b are called leaves and the other vertices internal vertices, $|b|$ denotes the number of internal vertices of b .
2. We denote by \circ the planar tree without internal vertex.
3. A planar tree is a p -tree if each of its internal vertices has exactly p childrens. We denote by $\mathbb{T}(p)$ the set of p -trees.

Example 1.1

The planar trees of remark 1.1 are 2-trees and they satisfy $|b| = 2$.

Definition 1.2 Let $(b_1, \dots, b_p) \in \mathbb{T}(p)^p$, then we denote by $B_+(b_1, \dots, b_p)$ the p -tree obtained by connecting to a new root the roots of b_1 and b_2 and \dots and b_p .

$$B_+(b_1, \dots, b_p) =$$

Property 1.1 Let $b \in \mathbb{T}$ be such that $b \neq \circ$ then there exists a unique p -uplet $(b_1, \dots, b_p) \in \mathbb{T}(p)^p$ such that $b = B_+(b_1, \dots, b_p)$.

2 Perturbative classical field theory: Butcher series

Let q be an integer such that $q > d/2$. Then it is well known (see e.g. [1]) that the Sobolev space $H^q(\mathbb{R}^d)$ is an algebra, more precisely we have

Proposition 2.1 Let f and g belong to $H^q(\mathbb{R}^d)$ then the product fg still belongs to $H^q(\mathbb{R}^d)$ and there exists a constant $c_q > 0$ which depends only on q and d such that

$$\|fg\|_{H^q(\mathbb{R}^d)} \leq c_q \|f\|_{H^q(\mathbb{R}^d)} \|g\|_{H^q(\mathbb{R}^d)}$$

From now on, for all $q \in \mathbb{N}$, H^q denotes the Sobolev space $H^q(\mathbb{R}^d)$.

Let $T > 0$ and $(\varphi^0, \varphi^1) \in H^{q+1} \times H^q$. Consider the following problem

$$\begin{cases} \varphi \in \mathcal{C}^1([0, T], H^q) \cap \mathcal{C}^2([0, T], H^{q-1}) \\ (\square + m^2)\varphi + \lambda\varphi^p = 0 \text{ in } H^{q-1} \\ \varphi(0, \bullet) = \varphi^0 ; \frac{\partial \varphi}{\partial t}(0, \bullet) = \varphi^1 \end{cases} \quad (2.1)$$

We define recursively the family $(\phi(b))_{b \in \mathbb{T}(p)}$ by setting

$$\widehat{\phi(\circ)}(t, \vec{k}) := \frac{\sin(\omega_k t)}{\omega_k} \widehat{\varphi^1}(\vec{k}) + \cos(\omega_k t) \widehat{\varphi^0}(\vec{k}) \quad (2.2)$$

where for all $\psi \in \mathcal{C}^0([0, T], \mathbb{R}^d)$, $\widehat{\psi}(t, \vec{k})$ denotes the spatial Fourier transform of ψ . For all $(b_1, \dots, b_p) \in \mathbb{T}(p)^p$ we set for all $x \in \mathbb{R}^n$

$$\phi(B_+(b_1, \dots, b_p))(x) := - \int_{P_+} G_{ret}(x - y) \phi(b_1)(y) \cdots \phi(b_p)(y) \quad (2.3)$$

where $P_+ := \{(t, \vec{x}) \in \mathbb{R}^n | t > 0\}$ and G_{ret} denotes the retarded Green function of the Klein-Gordon operator i.e.

$$G_{ret}(z) := \frac{1}{(2\pi)^d} \theta(z^0) \int_{\mathbb{R}^d} d^d k \frac{\sin(z^0 \omega_k)}{\omega_k} e^{ik \cdot \vec{z}} d^n y \quad (2.4)$$

θ denotes the heavyside function ($\theta(t) = 0$ if $t < 0$ and 1 otherwise).

Remark 2.1

We can easily check that $\phi(\circ)$ is the solution of problem (2.1) with $\lambda = 0$ i.e.

$$\begin{aligned} (\square + m^2)\phi(\circ) &= 0 \\ \phi(\circ)(0, \bullet) &= \varphi^0 ; \frac{\partial \phi(\circ)}{\partial t}(0, \bullet) = \varphi^1 \end{aligned}$$

So we can consider $\phi(\circ)$ as "the free field corresponding to the interacting field φ at time $t = 0$ ".

In other hand $\phi(B_+(b_1 \dots b_p))$ satisfies

$$(\square + m^2)\phi(B_+(b_1 \dots b_p)) = -\phi(b_1) \cdots \phi(b_p)$$

with zero Cauchy data on the hypersurface $t = 0$.

Then we have the following result

Theorem 2.1 For all $T > 0$ and $(\varphi^0, \varphi^1) \in H^{q+1} \times H^q$ the family $(\phi(b))_{b \in \mathbb{T}(p)}$ is well defined by (2.2) and (2.3) and $\forall b \in \mathbb{T}(p)$, $\phi(b)$ belongs to $\mathcal{C}^1([0, T], H^q) \cap \mathcal{C}^2([0, T], H^{q-1})$. Moreover there exists a constant $C > 0$ which depends only on m, d, p and q such that if

$$C|\lambda(1 + MT)|(\|\varphi^0\| + \|\varphi^1\|)^{p-1} < 1 \quad (2.5)$$

(here M denotes the constant $M := \max(m, 1/m)$) then the power series

$$\varphi = \sum_{b \in \mathbb{T}(p)} \lambda^{|b|} \phi(b)$$

converges in the $\mathcal{C}^1([0, T], H^q) \cap \mathcal{C}^2([0, T], H^{q-1})$ topology and the sum φ is a solution of problem (2.1).

Remarks 2.1

We have studied the equation (2.1) but our approach can be extended for analytic non linearity. In this case, all planar trees are involved and condition (2.5) must be adapted (see [9] for more details).

Proof: (of theorem 2.1)

The proof of theorem (2.1) is very simple. Suppose that the power series $\varphi = \sum_{b \in \mathbb{T}(p)} \lambda^{|b|} \phi(b)$ converges then a simple calculation show that we have $\varphi(0, \bullet) = \phi(\circ)(0, \bullet) + 0 = \varphi^0$ and $\frac{\partial \varphi}{\partial t}(0, \bullet) = \frac{\partial \phi(\circ)}{\partial t}(0, \bullet) + 0 = \varphi^1$ and for all $(b_1 \dots b_p) \in \mathbb{T}(p)^p$

$$(\square + m^2)\phi(B_+(b_1, \dots, b_p)) = -\phi(b_1) \dots \phi(b_p)$$

But for all $b \in \mathbb{T}(p)$, $b \neq \circ$ there exists a unique p -uplet $(b_1 \dots b_p)$ of p -tree such that $b = B_+(b_1 \dots b_p)$ hence we get

$$(\square + m^2)\varphi = (\square + m^2)\phi(\circ) + \sum_{(b_1, \dots, b_p) \in \mathbb{T}(p)^p} \lambda^{|B_+(b_1, \dots, b_p)|} (\square + m^2)\phi(B_+(b_1, \dots, b_p)) \quad (2.6)$$

$$= 0 - \lambda \sum_{(b_1, \dots, b_p) \in \mathbb{T}(p)^p} \lambda^{|b_1| + \dots + |b_p|} \phi(b_1) \dots \phi(b_p) = -\lambda \varphi^p \quad (2.7)$$

Let us focus on the convergence of the power series. Let us show recursively that $\phi(b)$ belongs to $\mathcal{C}^1([0, T], H^q) \cap \mathcal{C}^2([0, T], H^{q-1})$ for all $b \in \mathbb{T}(p)$ and

$$\|\phi(b)\| \leq (c_q^{p-1}(1 + MT))^{|b|} [M^2(\|\varphi^0\| + \|\varphi^1\|)]^{|b|(p-1)+1} \quad (2.8)$$

where M denotes the constant $M := \max(m, 1/m) \geq 1$.

Let us show (2.8) for $b = \circ$. The function $\phi(\circ)$ is given by (2.2). Then we can easily show that $\phi(\circ)$ belong to $\mathcal{C}^1([0, T], H^q) \cap \mathcal{C}^2([0, T], H^{q-1})$ and since $(m^2 + \alpha^2)/(1 + \alpha^2) \leq M^2$ for all $\alpha \in \mathbb{R}$ we have

$$\begin{aligned} \|\phi(\circ)(t, \bullet)\|_{H^q} &\leq M\|\varphi^1\|_{H^q} + \|\varphi^0\|_{H^q} \leq M^2(\|\varphi^0\| + \|\varphi^1\|) \\ \left\| \frac{\partial \phi(\circ)}{\partial t}(t, \bullet) \right\|_{H^q} &\leq \|\varphi^1\|_{H^q} + M\|\varphi^0\|_{H^{q+1}} \leq M^2(\|\varphi^0\| + \|\varphi^1\|) \\ \left\| \frac{\partial^2 \phi(\circ)}{\partial t^2}(t, \bullet) \right\|_{H^{q-1}} &\leq M\|\varphi^1\|_{H^q} + M^2\|\varphi^0\|_{H^{q+1}} \leq M^2(\|\varphi^0\| + \|\varphi^1\|) \end{aligned}$$

so $\|\phi(\circ)\| := \max_{t \in [0, T]} \left(\|\phi(\circ)(t, \bullet)\|_{H^q}; \left\| \frac{\partial \phi(\circ)}{\partial t}(t, \bullet) \right\|_{H^q}; \left\| \frac{\partial^2 \phi(\circ)}{\partial t^2}(t, \bullet) \right\|_{H^{q-1}} \right)$ satisfies (2.8).

Suppose that (2.8) is satisfied for all $b \in \mathbb{T}(p)$, $|b| \leq N$. Let $b \in \mathbb{T}(p)$ such that $|b| = N + 1 \geq 1$ then $\exists (b_1 \dots b_p) \in \mathbb{T}(p)^p$ such that $b = B_+(b_1 \dots b_p)$ and $\phi(b)$ is defined by (2.3) hence, using proposition 2.1 we get

$$\begin{aligned} \|\phi(b)(t, \bullet)\|_{H^q} &\leq c_q^{p-1} MT \|\phi(b_1)\| \cdots \|\phi(b_p)\| \leq (1 + MT) c_q^{p-1} \|\phi(b_1)\| \cdots \|\phi(b_p)\| \\ \left\| \frac{\partial \phi(b)}{\partial t}(t, \bullet) \right\|_{H^q} &\leq c_q^{p-1} T \|\phi(b_1)\| \cdots \|\phi(b_p)\| \leq (1 + MT) c_q^{p-1} \|\phi(b_1)\| \cdots \|\phi(b_p)\| \\ \left\| \frac{\partial^2 \phi(b)}{\partial t^2}(t, \bullet) \right\|_{H^{q-1}} &\leq c_q^{p-1} (1 + MT) \|\phi(b_1)\| \cdots \|\phi(b_p)\| \end{aligned}$$

hence we have $\|\phi(b)\| \leq (1 + MT) c_q^{p-1} \|\phi(b_1)\| \cdots \|\phi(b_p)\|$. Since (2.8) is satisfied for b_1, \dots, b_p and using the fact that $|B_+(b_1 \dots b_p)| = |b_1| + \dots + |b_p| + 1$ we finally get (2.8) for b .

Finally it can be shown (see e.g. [16]) that the number of p -trees b such that $|b| = N$ is bounded by $(p^p / (p-1)^{p-1})^N$ hence we see that if we have

$$(1 + MT) |\lambda| (\|\varphi^0\| + \|\varphi^1\|)^{p-1} < \frac{(p-1)^{p-1}}{p^p c_q^{p-1} M^{2(p-1)}} \quad (2.9)$$

then the power series $\sum_N |\lambda|^N \|\phi(b)\|$ converges which completes the proof. ■

Remark 2.2

Notice that the proof gives an explicit value for C .

3 From Butcher series to perturbative quantum field theory

We have seen that Butcher series provides an explicite and precise perturbative expansion of the *solutions* of interacting Klein–Gordon equation

$$(\square + m^2)\varphi + \lambda\varphi^p = 0 \quad (E_p)$$

So we get a perturbative *classical* field theory.

On the other hand, physicists devolopped a perturbative quantum field theory (see e.g. [12], [10], [15]). The question we are interesting in is the following: is there a link between Butcher series and perturbative quantum field theory ?

The answer is yes, we will show that the Heisenberg picture of interacting quantum field can be written as a quantized Butcher series.

Let $t_0 \in \mathbb{R}$ be a fixed time and $x = (t, \vec{x}) \in \mathbb{R}^{1,d}$. Then following M. Peskin et D. Schroeder [12] p. 83, the free field on space–time is the self–adjoint operator $\phi_I(x)$ acting on *Fock space* (see [14], [13]) defined by

$$\phi_I(x) = \phi_I(t, \vec{x}) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{dk}{\sqrt{2\omega_k}} \left(\mathbf{a}_k e^{-ik \cdot x} + \mathbf{a}_k^\dagger e^{ik \cdot x} \right) \Big|_{x^0=t-t_0}$$

Here $k \cdot x$ denotes the quantity $k \cdot x := \omega_k t - \langle k, \vec{x} \rangle$ where $\langle k, \vec{x} \rangle$ denotes the usual scalar product on \mathbb{R}^d and ω_k the quantity $\omega_k := (|k|^2 + m^2)^{1/2}$. The operator \mathbf{a}_k and \mathbf{a}_k^\dagger denote the usual *creation* and *annihilation* operator (see e.g. [13]).

We formally define the family $(\hat{\phi}(b)(x))_{b \in \mathbb{T}(p)}$ of operators by the following

Definition 3.1 $\widehat{\phi}(\circ)(x) := \phi_I(x)$ and for all $(b_1, \dots, b_p) \in \mathbb{T}(p)^p$

$$\widehat{\phi}(B_+(b_1, \dots, b_p))(t, \vec{x}) := - \int_{t_0}^t dy^0 \int_{\mathbb{R}^d} d\vec{y} G_{ret}(x-y) \widehat{\phi}(b_1)(y) \cdots \widehat{\phi}(b_p)(y) \quad (3.1)$$

Hence we can see the sum $\sum_{b \in \mathbb{T}(p)} \lambda^{|b|} \widehat{\phi}(b)(x)$ as the formal quantization of the Butcher series $\sum_{b \in \mathbb{T}(p)} \lambda^{|b|} \phi(b)$.

Remark 3.1

- The definition 3.1 is formal because we do not care about the definition of the operator product which is well known to be ill defined.
- We have the following commutation relation between the operators $\phi_I(x)$ and $\phi_I(y)$

$$\Delta(x-y) := [\phi_I(x), \phi_I(y)] = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{dk}{2\omega_k} \left(e^{-ik \cdot (x-y)} - e^{ik \cdot (x-y)} \right) \quad (3.2)$$

(see [12] p.28). Using the expression (2.4) of the retarded Green function G_{ret} , we get

$$G_{ret}(z) = -i\theta(\tau)\Delta(z) \quad (3.3)$$

This simple remark leads to the following theorem:

Theorem 3.1 for all $t > t_0$ and $\vec{x} \in \mathbb{R}^d$ we have

$$\widehat{\varphi}(t, \vec{x}) := \sum_{b \in \mathbb{T}(p)} \lambda^{|b|} \widehat{\phi}(b)(x) = U^\dagger(t) \phi_I(t, \vec{x}) U(t) \quad (3.4)$$

where $U(t)$ denotes the operator

$$U(t) := \sum_{\alpha \geq 0} (i\lambda)^\alpha \int_{t_0}^t d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \cdots \int_{t_0}^{\tau_{\alpha-1}} d\tau_\alpha H_I(\tau_1) H_I(\tau_2) \cdots H_I(\tau_\alpha) \quad (3.5)$$

and $H_I(\tau)$ is given by

$$H_I(\tau) := \frac{1}{p+1} \int_{\mathbb{R}^d} d\vec{y} \phi_I^{p+1}(\tau, \vec{y})$$

Remarks 3.1

- Using the time ordered product (see [12] p.85 for a definition) the operator U is given by the well known formula

$$U(t) = T \left\{ \exp \left(i\lambda \int_{t_0}^t d\tau H_I(\tau) \right) \right\}$$

- The right hand side of (3.4) is exactly the Heisenberg picture of interacting quantum field and this can be the beginning of the perturbative quantum field theory (see e.g. [12] p.77-87).

Proof: (of theorem 3.1)

Let us introduce a new notation which will be very helpful for manipulations of iterated integrals such as (3.5). Let $r \in \mathbb{N}^*$ and $t_0 \leq t$. Then the symbol $\oint_{t_0}^t dy_1 \cdots dy_r$ denotes the integrals

$$\oint_{t_0}^t dy_1 \cdots dy_r := \int_{t_0}^t dy_1^0 \int_{t_0}^{\tau_1} dy_2^0 \cdots \int_{t_0}^{\tau_{p-1}} dy_p^0 \int_{\mathbb{R}^d} d^d \vec{y}_1 \cdots \int_{\mathbb{R}^d} d^d \vec{y}_r$$

Hence using this notation, expression (3.5) becomes

$$U(t) = \sum_{\alpha \geq 0} \left(\frac{i\lambda}{p+1} \right)^\alpha \int_{t_0}^t dy_1 \cdots dy_\alpha \phi_I^{p+1}(y_1) \phi_I^{p+1}(y_2) \cdots \phi_I^{p+1}(y_\alpha) \quad (3.6)$$

So since $\phi_I(x)$ is self-adjoint we get the following identity

$$\begin{aligned} U^\dagger(t) \phi_I(t, \vec{x}) U(t) &= \sum_{m \geq 0} \left(\frac{i\lambda}{p+1} \right)^m \sum_{\substack{(r,s) \in \mathbb{N} \\ r+s=m}} \\ &(-1)^r \int_{t_0}^t dy_1 \cdots dy_r \int_{t_0}^t dz_1 \cdots dz_s \phi_I^{p+1}(y_r) \cdots \phi_I^{p+1}(y_1) \phi_I(x) \phi_I^{p+1}(z_1) \cdots \phi_I^{p+1}(z_s) \end{aligned}$$

To complete the proof of theorem 3.1, it suffices to show that for all $m \in \mathbb{N}$

$$\begin{aligned} \sum_{\substack{b \in \mathbb{T}(p) \\ |b|=m}} \widehat{\phi}(b)(x) &= \left(\frac{i}{p+1} \right)^m \sum_{\substack{(r,s) \in \mathbb{N} \\ r+s=m}} (-1)^r \int_{t_0}^t dy_1 \cdots dy_r \int_{t_0}^t dz_1 \cdots dz_s \\ &\phi_I^{p+1}(y_r) \cdots \phi_I^{p+1}(y_1) \phi_I(x) \phi_I^{p+1}(z_1) \cdots \phi_I^{p+1}(z_s) \quad (3.7) \end{aligned}$$

Let us prove (3.7) by induction on $m \in \mathbb{N}$.

For $m = 0$ identity (3.7) is obvious since the left hand side of (3.7) reduces to $\phi_I(x)$ and the only p -tree such that $|b| = 0$ is given by $b = \circ$ therefore the right hand side equals to $\widehat{\phi}(\circ)(x) = \phi_I(x)$.

Let $N \in \mathbb{N}$ and suppose that identity (3.7) is satisfied for all $m \leq N$. Then set

$$\varphi_{N+1}(x) := \sum_{\substack{b \in \mathbb{T}(p) \\ |b|=N+1}} \widehat{\phi}(b)(x)$$

Since $N+1 \geq 1$, for all $b \in \mathbb{T}(p)$ such that $|b| = N+1$ there is a unique $(b_1, \dots, b_p) \in \mathbb{T}(p)^p$ such that $b = B_+(b_1, \dots, b_p)$. So the definition of $\widehat{\phi}(b)(x)$ together with remark 3.1 lead to

$$\varphi_{N+1} = \sum_{\substack{(b_1, \dots, b_p) \in \mathbb{T}(p)^p \\ |b_1| + \dots + |b_p| = N}} i \int_{t_0}^t dy^0 \int_{\mathbb{R}^d} d\vec{y} \Delta(x-y) \widehat{\phi}(b_1)(y) \cdots \widehat{\phi}(b_p)(y)$$

which can be rewritten as

$$\varphi_{N+1} = \sum_{\substack{(q_1, \dots, q_p) \in \mathbb{N}^p \\ q_1 + \dots + q_p = N}} i \int_{t_0}^t dy^0 \int_{\mathbb{R}^d} d\vec{y} \Delta(x-y) \left(\sum_{\substack{b_1 \in \mathbb{T}(p) \\ |b_1|=q_1}} \widehat{\phi}(b_1)(y) \right) \cdots \left(\sum_{\substack{b_p \in \mathbb{T}(p) \\ |b_p|=q_p}} \widehat{\phi}(b_p)(y) \right).$$

Then using (3.7) for $m \leq N$, we see that φ_{N+1} is given by the following expression

$$\begin{aligned} i \left(\frac{i}{p+1} \right)^N \sum_{\substack{(q_1, \dots, q_p) \in \mathbb{N}^p \\ q_1 + \dots + q_p = N}} \sum_{\substack{(r_1, s_1) \in \mathbb{N}^2 \\ r_1 + s_1 = q_1}} \cdots \sum_{\substack{(r_p, s_p) \in \mathbb{N}^2 \\ r_p + s_p = q_p}} (-1)^{r_1 + \dots + r_p} \\ \int_{t_0}^t dy^0 \int_{\mathbb{R}^d} d\vec{y} \Delta(x-y) \int_{t_0}^{y^0} dy_{1,r_1}^{(1)} \int_{t_0}^{y^0} dz_{1,s_1}^{(1)} \cdots \int_{t_0}^{y^0} dy_{1,r_p}^{(p)} \int_{t_0}^{y^0} dz_{1,s_p}^{(p)} \mathbb{P}((y_j^{(k)})_{j,k}, y) \quad (3.8) \end{aligned}$$

Here we must explain the notations:

- Let α, β be some integers such that $\alpha \leq \beta$, then we denote by $y_{\alpha, \beta}$ the $(\beta - \alpha + 1)$ -uplet $y_{\alpha, \beta} = (y_\alpha, \dots, y_\beta) \in (\mathbb{R}^n)^{\beta - \alpha + 1}$.
- $\mathbb{P}((y_j^{(k)})_{j,k}, y)$ denotes the product

$$\begin{aligned} & \phi_I^{p+1}(y_{r_1}^{(1)}) \cdots \phi_I^{p+1}(y_1^{(1)}) \phi_I(y) \left[\phi_I^{p+1}(z_1^{(1)}) \cdots \phi_I^{p+1}(z_{s_1}^{(1)}) \phi_I^{p+1}(y_{r_2}^{(2)}) \cdots \phi_I^{p+1}(y_1^{(2)}) \right] \phi_I(y) \\ & \left[\phi_I^{p+1}(z_1^{(2)}) \cdots \phi_I^{p+1}(z_{s_2}^{(2)}) \phi_I^{p+1}(z_{r_3}^{(3)}) \cdots \phi_I^{p+1}(z_1^{(3)}) \right] \phi_I(y) \cdots \phi_I(y) \phi_I^{p+1}(z_1^{(p)}) \cdots \phi_I^{p+1}(z_{s_p}^{(p)}) \end{aligned}$$

Then we use the well known combinatorial lemma of quantum field theory (see e.g. [12]) which can be proved directly using iterated integrals

Lemma 3.1 For all $t \geq 0$ we have $U(t)U^\dagger(t) = Id$ i.e. for all $m \in \mathbb{N}$

$$\sum_{\substack{(r,s) \in \mathbb{N}^2 \\ r+s=m}} (-1)^r \int_{t_0}^t dy'_1 \cdots dy'_r \int_{t_0}^t dy_1 \cdots dy_s \phi_I^{p+1}(y_1) \cdots \phi_I^{p+1}(y_s) \phi_I^{p+1}(y'_r) \cdots \phi_I^{p+1}(y'_1) = \delta_{0,m}$$

where $\delta_{0,m} = 0$ if $m \neq 0$ and 1 otherwise.

Using lemma 3.1 we see that expression (3.8) reduces to

$$\begin{aligned} \varphi_{N+1} &= i \left(\frac{i}{p+1} \right)^N \sum_{\substack{(r,s) \in \mathbb{N}^2 \\ r+s=N}} (-1)^r \int_{t_0}^t dy^0 \int_{\mathbb{R}^d} d\vec{y} \Delta(x-y) \\ & \quad \int_{t_0}^{y^0} dy_1 \cdots dy_r \int_{t_0}^{y^0} dz_1 \cdots dz_s \phi_I^{p+1}(y_r) \cdots \phi_I^{p+1}(y_1) \phi_I^p(y) \phi_I^{p+1}(z_1) \cdots \phi_I^{p+1}(z_s) \quad (3.9) \end{aligned}$$

But since $\Delta(x-y) = [\phi_I(x), \phi_I(y)]$ which commutes with $\phi_I(z)$ (it is a c -number), we can replace $\Delta(x-y)\phi_I(y)^p$ in (3.9) by its expression using $\phi_I(x)$ and $\phi_I(y)$ i.e.

$$\Delta(x-y)\phi_I(y)^p = \phi_I(x)\phi_I^{p+1}(y) - \phi_I(y)\phi_I(x)\phi_I^p(y).$$

But we have $\phi_I(x)\phi_I^p(y) = \phi_I^p(y)\phi_I(x) + p\Delta(x-y)\phi_I^{p-1}(y)$ therefore the last identity leads to

$$\Delta(x-y)\phi_I(y)^p = \phi_I(x)\phi_I^{p+1}(y) - \phi_I^{p+1}(y)\phi_I(x) - p\Delta(x-y)\phi_I^p(y).$$

Inserting this last expression in (3.9) we finally get

$$\begin{aligned} \varphi_{N+1} &= \left(\frac{i}{p+1} \right)^{N+1} \sum_{\substack{(r,s) \in \mathbb{N}^2 \\ r+s=N}} (-1)^r \int_{t_0}^t dy^0 \int_{\mathbb{R}^d} d\vec{y} \int_{t_0}^{y^0} dy_1 \cdots dy_r \int_{t_0}^{y^0} dz_1 \cdots dz_s \\ & \quad \left[\phi_I^{p+1}(y_r) \cdots \phi_I^{p+1}(y_1) \phi_I(x) \phi_I^{p+1}(y) \phi_I^{p+1}(z_1) \cdots \phi_I^{p+1}(z_s) \right. \\ & \quad \left. - \phi_I^{p+1}(y_r) \cdots \phi_I^{p+1}(y_1) \phi_I^{p+1}(y) \phi_I(x) \phi_I^{p+1}(z_1) \cdots \phi_I^{p+1}(z_s) \right] \quad (3.10) \end{aligned}$$

Consider separately the terms of the right hand side of (3.10). The first term is given by (modulo a factor $(i/(p+1))^{N+1}$)

$$\begin{aligned} & \sum_{\substack{(r,s) \in \mathbb{N}^2 \\ r+s=N}} (-1)^r \int_{t_0}^t dy^0 \int_{\mathbb{R}^d} d\vec{y} \int_{t_0}^{y^0} dy_1 \cdots dy_r \int_{t_0}^{y^0} dz_1 \cdots dz_s \\ & \quad \phi_I^{p+1}(y_r) \cdots \phi_I^{p+1}(y_1) \phi_I(x) \phi_I^{p+1}(y) \phi_I^{p+1}(z_1) \cdots \phi_I^{p+1}(z_s) \quad (3.11) \end{aligned}$$

Notice that for all $y \in [0, t] \times \mathbb{R}^d$ and $a \geq 1$ we have

$$\oint_{t_0}^{y^0} dy_1 \cdots dy_a = \oint_{t_0}^t dy_1 \cdots dy_a - \int_{y^0}^t dy_1^0 \int_{\mathbb{R}^d} d\vec{y}_1 \oint_{t_0}^{y_1^0} dy_2 \cdots dy_a$$

Hence using this last identity and performing the change of variable $s \leftarrow s + 1$, expression (3.11) leads to

$$\begin{aligned} & \sum_{\substack{r \geq 0; s \geq 1 \\ r+s=N+1}} (-1)^s \oint_{t_0}^t dy_1 \cdots dy_r \oint_{t_0}^t dz_1 \cdots dz_s \mathbb{V}(y_r \dots y_1, x, z_1 \dots z_s) \\ & - \sum_{\substack{r \geq 1, s \geq 1 \\ r+s=N+1}} (-1)^r \int_{t_0}^t dz_1^0 \int_{\mathbb{R}^d} d\vec{z}_1 \int_{z_1^0}^t dy_1^0 \int_{\mathbb{R}^d} d\vec{y}_1 \\ & \quad \oint_{t_0}^{y_1^0} dy_2 \cdots dy_r \oint_{t_0}^{z_1^0} dz_2 \cdots dz_s \mathbb{V}(y_r \dots y_1, x, z_1 \dots z_s) \quad (3.12) \end{aligned}$$

where $\mathbb{V}(y_r \dots y_1, x, z_1 \dots z_s)$ denotes the product

$$\mathbb{V}(y_r \dots y_1, x, z_1 \dots z_s) := \phi_I^{p+1}(y_r) \cdots \phi_I^{p+1}(y_1) \phi_I(x) \phi_I^{p+1}(z_1) \cdots \phi_I^{p+1}(z_s)$$

A similar computation (only replacing y by z and r by s) shows that the second term of the right hand side of (3.10) writes (modulo a factor $-i \left(\frac{i}{p+1}\right)^N$):

$$\begin{aligned} & \sum_{\substack{r \geq 1; s \geq 0 \\ r+s=N+1}} (-1)^s \oint_{t_0}^t dy_1 \cdots dy_r \oint_{t_0}^t dz_1 \cdots dz_s \mathbb{V}(y_r \dots y_1, x, z_1 \dots z_s) \\ & - \sum_{\substack{r \geq 1, s \geq 1 \\ r+s=N+1}} (-1)^r \int_{t_0}^t dy_1^0 \int_{\mathbb{R}^d} d\vec{y}_1 \int_{y_1^0}^t dz_1^0 \int_{\mathbb{R}^d} d\vec{z}_1 \\ & \quad \oint_{t_0}^{y_1^0} dy_2 \cdots dy_r \oint_{t_0}^{z_1^0} dz_2 \cdots dz_s \mathbb{V}(y_r \dots y_1, x, z_1 \dots z_s) \quad (3.13) \end{aligned}$$

Hence injecting (3.12) and (3.13) in (3.10), we finally find out that operator $\left(\frac{p+1}{i}\right)^{N+1} \varphi_{N+1}$ is given by

$$\begin{aligned} & 2 \sum_{\substack{r \geq 1; s \geq 1 \\ r+s=N+1}} (-1)^s \oint_{t_0}^t dy_1 \cdots dy_r \oint_{t_0}^t dz_1 \cdots dz_s \mathbb{V}(y_r \dots y_1, x, z_1 \dots z_s) \\ & + (-1)^{N+1} \oint_{t_0}^t dy_1 \cdots dy_{N+1} \mathbb{V}(y_{N+1} \dots y_1, x) + \oint_{t_0}^t dz_1 \cdots dz_{N+1} \mathbb{V}(x, z_1 \dots z_{N+1}) + A \quad (3.14) \end{aligned}$$

where A denotes the operator

$$\begin{aligned} & - \sum_{\substack{r \geq 1; s \geq 1 \\ r+s=N+1}} (-1)^s \int_{\mathbb{R}^d} d\vec{y}_1 \int_{\mathbb{R}^d} d\vec{z}_1 \left(\int_{t_0}^t dz_1^0 \int_{z_1^0}^t dy_1^0 + \int_{t_0}^t dy_1^0 \int_{y_1^0}^t dz_1^0 \right) \\ & \quad \oint_{t_0}^{y_1^0} dy_2 \cdots dy_r \oint_{t_0}^{z_1^0} dz_2 \cdots dz_s \mathbb{V}(y_r \dots y_1, x, z_1 \dots z_s) \quad (3.15) \end{aligned}$$

But we have the following identity

$$\int_{t_0}^t dz_1^0 \int_{z_1^0}^t dy_1^0 + \int_{t_0}^t dy_1^0 \int_{y_1^0}^t dz_1^0 = \int_{t_0}^t dz_1^0 \int_{t_0}^t dy_1^0$$

So inserting this last identity in the expression (3.15) of A we get

$$A = - \sum_{\substack{r \geq 1; s \geq 1 \\ r+s=N+1}} (-1)^s \oint_{t_0}^t dy_1 \cdots dy_r \oint_{t_0}^t dz_1 \cdots dz_s \mathbb{V}(y_r \dots y_1, x, z_1 \dots z_s)$$

hence we finally see that (3.14) leads to

$$\left(\frac{p+1}{i}\right)^{N+1} \varphi_{N+1} = \sum_{\substack{r \geq 0; s \geq 0 \\ r+s=N+1}} (-1)^s \oint_{t_0}^t dy_1 \cdots dy_r \oint_{t_0}^t dz_1 \cdots dz_s \mathbb{V}(y_r \dots y_1, x, z_1 \dots z_s)$$

which is exactly (3.7) at order $N + 1$. ■

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